A note on leapfrogging vortex rings

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Abstract. In this paper we provide examples, by numerical simulation using the Navier–Stokes equations for axisymmetric laminar flow, of the 'leapfrogging' motion of two, initially identical, vortex rings which share a common axis of symmetry. We show that the number of clear passes that each ring makes through the other increases with Reynolds number, and that as long as the configuration remains stable the two rings ultimately merge to form a single vortex ring.

1. Introduction

The long-held fascination for vortex rings by fluid-dynamicists is well documented. For example Acheson (1990) quotes extensively from correspondence between Kelvin and Helmholtz, and from the latter's original paper of 1858, whilst a recent review article by Shariff and Leonard (1992) demonstrates the ongoing interest in vortex rings. On the basis of the theorems he had himself devised, Helmholtz deduced that when two initially identical vortex rings travel in the same direction '. . . the foremost widens and travels more slowly, the pursuer shrinks and travels faster, till finally, if their velocities are not too different, it overtakes the first and penetrates it. Then the same game goes on in the opposite order, so that the rings pass through each other alternately'. In other words the vortex rings perform a leapfrogging motion.

A leapfrogging motion as described by Helmholtz can be realised in an inviscid fluid using a model of the type adopted, for example, by Dyson (1893) in which the cross-section of the vortex ring is small in diameter compared with the ring diameter, and is not allowed to deform from its circular shape. In that case leapfrogging can continue indefinitely. However the inviscid-fluid assumption itself is not sufficient to guarantee this. Shariff et al. (1988, 1989) have adopted a contour dynamics approach to these problems, which does allow for deformation of the vortex core. For the thinnest cores that were adopted, continued leapfrogging was established with quasi-periodic core deformation. For thicker cores, what they describe as a resonance phenomenon takes place in which during each passage the aspect ratio of the initially rear vortex increases; whilst for the thickest cores of all there was an ever-increasing elongation of the rearward vortex ring during its very first pass through the forward ring. Of course, in real fluids viscous diffusion may be expected to modify the vortex-ring structure. Indeed, in early experiments Maxworthy (1972) did not observe a clear passage of one ring through the other, and opined that '. . . contrary to popular belief, rings do not pass back and forth through one another, but that the rearward one becomes entrained into the forward one' (by viscous effects). However, subsequently, Yamada and Matsui (1978), see also Van Dyke (1982), were able to demonstrate a clear passage of one vortex ring through another with the rings appearing to merge only after a second passage. Maxworthy (1979) then correctly observed what is perhaps obvious with

hindsight, namely that there is a continuous variation of phenomena displayed by the vortex rings with Reynolds number. Indeed this is one of the main conclusions that can be drawn from the present work.

Our interest in the behaviour of co-axial vortex rings originates with the work of Riley and Weidman (1993), whose numerical and experimental work was directed at the following problem. If two vortex rings of different diameter, but with a common axis, have circulation of opposite sign, and lie initially in the same plane, is it possible for the two rings to remain co-planar, in the form of a 'vortex-ring pair', in the subsequent motion? An affirmative answer to this question was supplied, at least in the sense that a quasi-steady vortex-ring pair can propagate over several ring diameters. In a subsequent paper, Riley (1993) has exploited the numerical technique for the Navier–Stokes equations of Riley and Weidman to examine the behaviour of pairs of co-axial vortex rings which have circulation of opposite sign. The absolute values of the circulation for each ring may be different, as may their initial diameters, resulting in a rich and varied behaviour of the vortex rings.

In the present paper we exploit that same numerical solution technique for the Navier–Stokes equations, to discuss the behaviour of identical co-axial vortex rings, initially in tandem. We present results from several different configurations, with a detailed discussion of one case only, and relate our findings to earlier numerical investigations for both viscous and inviscid fluids, and experiments. The vortex rings are assumed to be devoid of swirl. We conclude this section with a brief account of the numerical approach, further details may be found in Riley and Weidman (1993).

Helmholtz's equations for the vorticity ω' may be written as

$$\frac{\partial \omega'}{\partial t} - \nabla \wedge (v' \wedge \omega') = - v\nabla \wedge \nabla \wedge \omega', \tag{1}$$

where v is the kinematic viscosity, v' the velocity, t' time and

$$\omega' = \nabla \wedge v', \qquad \nabla \cdot v' = 0. \tag{2a, b}$$

Our numerical simulations of (1) and (2) are carried out in a finite circular cylindrical container of radius a and length l, using cylindrical polar coordinates (r', θ, z') with $v' = (u', v', w')$. If γ_0 is the initial circulation about each vortex ring then to make our equations dimensionless we choose as a typical length a, time a^2/γ_0, velocity γ_0/a, and vorticity γ_0/a^2. For axi-symmetric flow we have the dimensionless velocity $v = (u, 0, w)$ and vorticity $\omega = (0, \zeta, 0)$ where $\zeta = \partial u/\partial z - \partial w/\partial r$. To satisfy (2b) we introduce a streamfunction ψ such that

$$u = \frac{1}{r}\frac{\partial \psi}{\partial z}, \qquad w = -\frac{1}{r}\frac{\partial \psi}{\partial r},$$

and we choose to work with a vorticity function $\Gamma = - r\zeta$, rather than ζ itself, so that

$$\Gamma = r\left(\frac{\partial w}{\partial r} - \frac{\partial u}{\partial z}\right).$$

The equations satisfied by Γ and ψ are, from the θ-components of (1) and (2a),

$$\left.\begin{array}{r}\dfrac{\partial \Gamma}{\partial t} + u\dfrac{\partial \Gamma}{\partial r} + w\dfrac{\partial \Gamma}{\partial z} - \dfrac{2u\Gamma}{r} = \dfrac{1}{\mathrm{Re}}\,\mathrm{D}^2\Gamma, \\ \mathrm{D}^2\psi = - \Gamma, \end{array}\right\} \tag{3a, b}$$
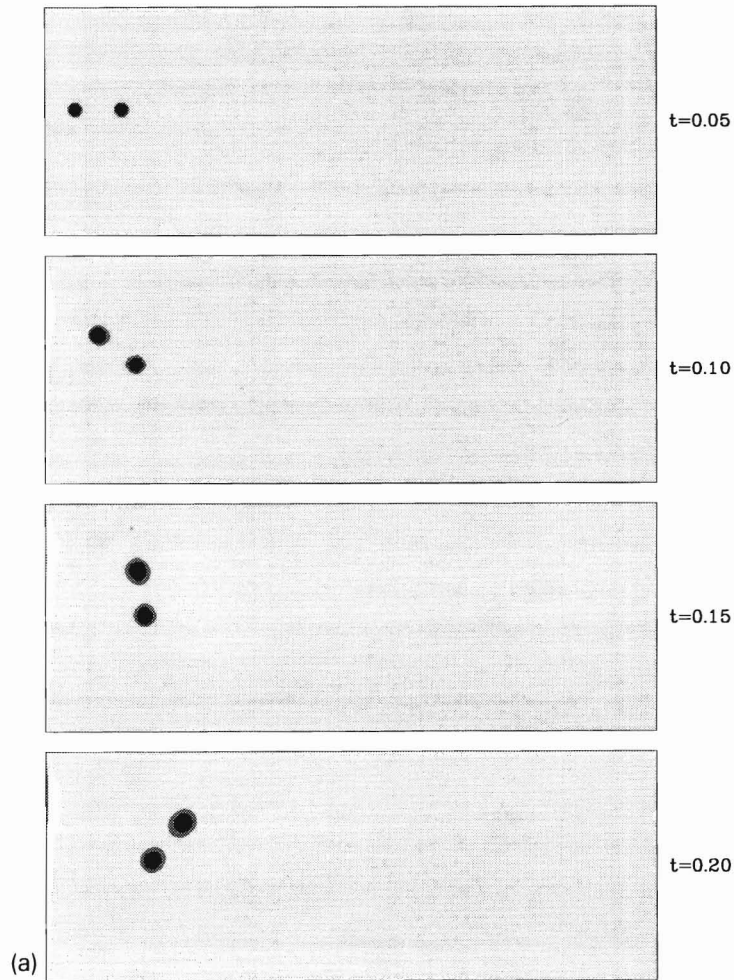
Fig. 1. The time history for a pair of vortex rings with $Re = 2000$, $t_0 = 0.05$, $r_0 = 0.34$, $z_1 = 0.1$, $z_2 = 0.25$. The lower cross-section of the two vortex rings is shown, so that the upper boundary of each time-frame represents the axis of symmetry. The lower boundary is at $r = 0.75$, and the two vertical boundaries at $z = 0, 2.0$ represent the end-walls of the computational domain. Black denotes a region in which $\zeta \geqslant 100$, dark grey a region within which $10 < \zeta < 100$, light grey a region for which $|\zeta| \leqslant 10$ and white for which $\zeta < -10$.

where $D^2 = \partial^2/\partial r^2 - r^{-1}\partial/\partial r + \partial^2/\partial z^2$, and $Re = \gamma_0/\nu$ is the Reynolds number. With $z_0 = l/a$ the boundary conditions which must be satisfied are:

$$
\left.
\begin{aligned}
\psi &= u = \partial w/\partial r = \Gamma = 0, & r &= 0, & 0 &\leqslant z \leqslant z_0, \\
\psi &= u = w = 0, \quad \Gamma = \partial w/\partial r, & r &= 1, & 0 &\leqslant z \leqslant z_0, \\
\psi &= u = w = 0, \quad \Gamma = -r\,\partial u/\partial z, & 0 &\leqslant r \leqslant 1, & z &= 0, z_0.
\end{aligned}
\right\}
\tag{4}
$$

The treatment of the conditions for Γ at $r = 1$, $z = 0, z_0$ follows that developed by Woods (1954), as described by Riley and Weidman (1993). With conditions (4) holding for all time t, we require an initial distribution of vorticity within the rings in order to initiate the calculation. At some instant, say t_0', vorticity in the core of the ring which was initially concentrated at $r' = r_0'$, will have diffused a distance $(\nu t_0')^{1/2}$. If this core size is small compared to the radius of the ring, so that
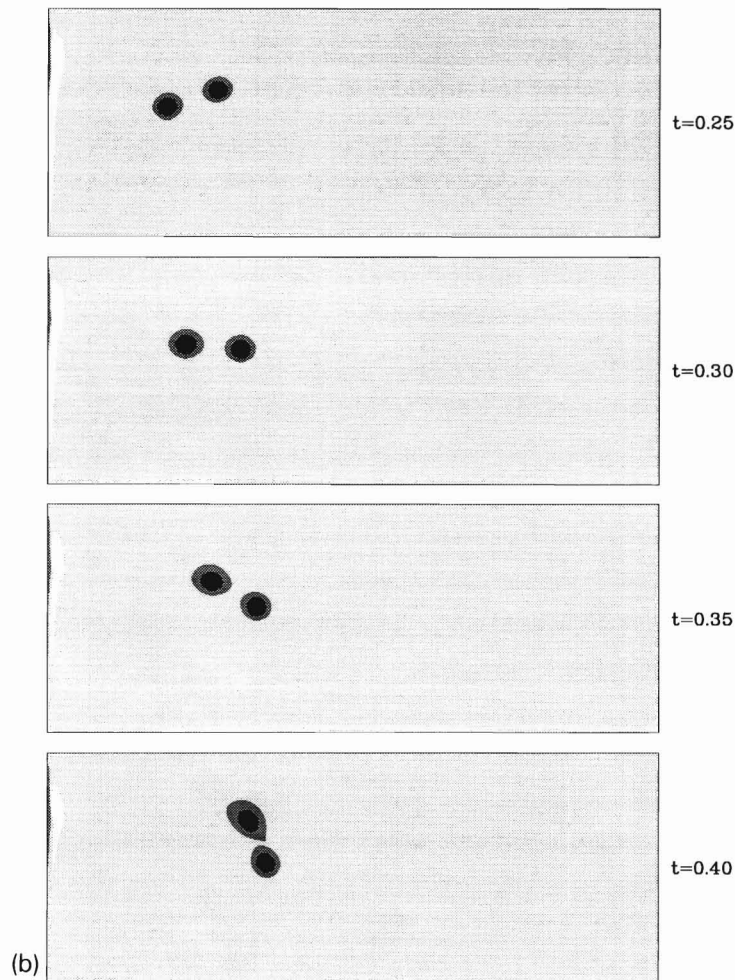
t=0.25

t=0.30

t=0.35

t=0.40

(b)

Fig. 1 (continued).

$(t_0/\mathrm{Re})^{1/2} \ll 1$, the vorticity may be approximated by the viscous line vortex solution of Lamb (1932). At $t = t_0$ then, the initial distribution for the vorticity function is

$$\Gamma = -\frac{r\,\mathrm{Re}}{4\pi t_0}\exp\{-\mathrm{Re}[(r-r_0)^2 + (z-z_1)^2]/4t_0\}$$

$$+ \exp\{-\mathrm{Re}[(r-r_0)^2 + (z-z_2)^2]/4t_0\}, \tag{5}$$

where z_1, z_2 are the axial positions of each ring at $t = t_0$; the initial distribution of ψ is then given from (3b).

We have solved the governing equations (3)–(5) using finite difference methods described in detail by Riley and Weidman, and given here in outline only. The solution of (3a) is determined by a standard ADI method in which advancement from t to $t + \delta t$ is achieved in two half-steps of $\frac{1}{2}\delta t$. With the solution for Γ determined at $t + \delta t$, the streamfunction, velocity components, and boundary conditions for Γ are all updated at that time level. This procedure is repeated over the same time interval until the solution is deemed to have converged according to some pre-set criterion. With spatial derivatives represented by central differences on a uniform computational
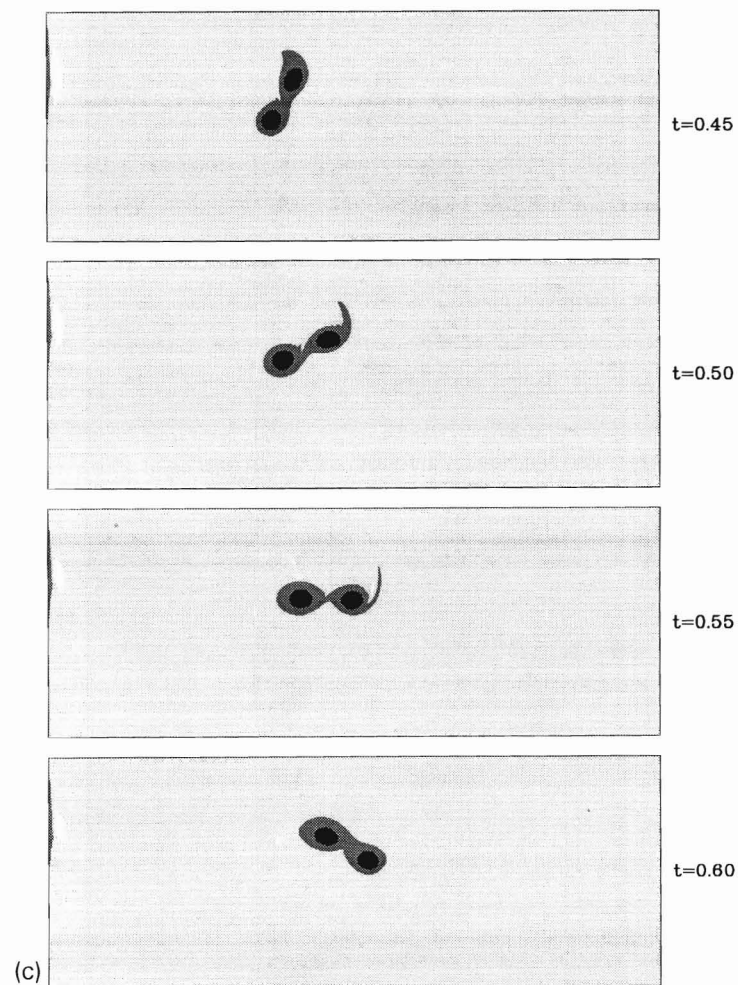
t=0.45

t=0.50

t=0.55

t=0.60

(c)

Fig. 1 (continued).

grid, the accuracy of our solution is, formally, to $O(\delta r^2, \delta z^2, \delta t^2)$. We describe, in the next section, solutions obtained by this method.

2. Calculations

As we have already remarked in section 1 there was no clear passage, in Maxworthy's experiments, of one ring through the other. By contrast, in the experiments of Yamada and Matsui it is only during the second passage that merging of the two rings begins, due to viscous diffusion. Evidently the latter experiments were carried out at a higher Reynolds number than the former. Stanaway et al. (1988) have attempted a simulation, from the Navier–Stokes equations of the experiments of Yamada and Matsui using a spectral method, and the results of their calculations are presented by Shariff and Leonard (1992). With $\mathrm{Re} = \gamma_0/\nu = 1000$ the two vortex rings merge following one clear passage of the rear ring through the forward. We have repeated Stanaway's calculation with $\mathrm{Re} = 1000$, with the initial configuration taken from fig. 6a of Shariff
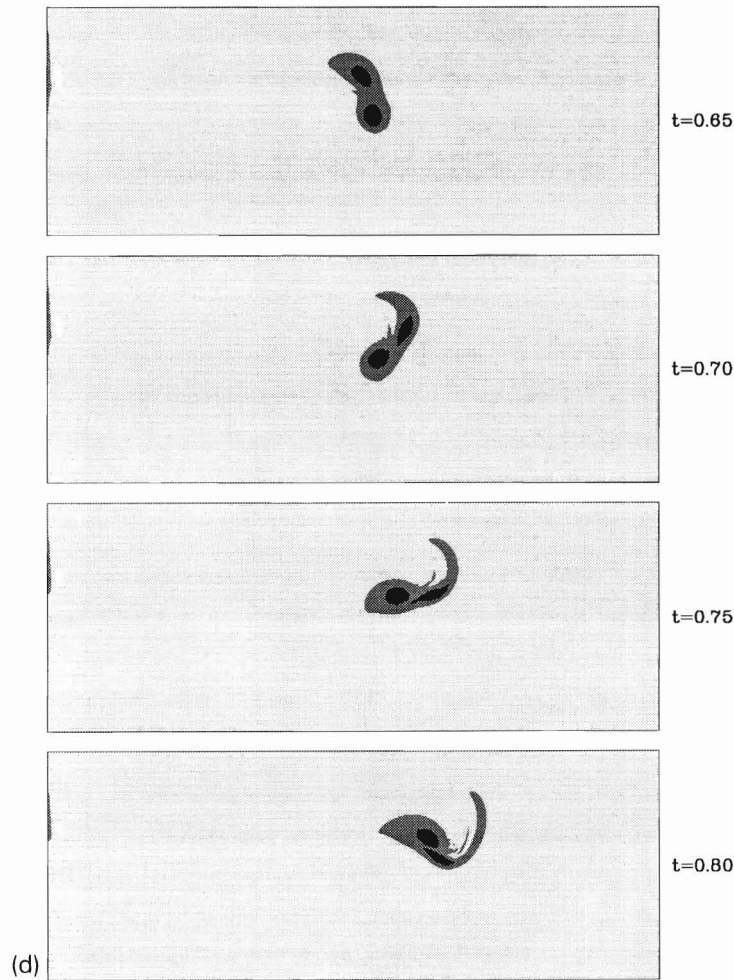
t=0.65

t=0.70

t=0.75

(d)

t=0.80

Fig. 1 (continued).

and Leonard's review article. This corresponds to $r_0 = 0.34$, $z_1 = 0.1$, $z_2 = 0.45$ and $t_0 = 0.205$ in eq. (5). The computational domain has $0 \leqslant r \leqslant 1$, $0 \leqslant z \leqslant 2$ with a mesh $\delta r = \delta z = 0.005$. $\delta t = 0.001$. We can confirm qualitative agreement between our results and Stanaway's as presented by Shariff and Leonard. As a second example the initial time t_0 was decreased to $t_0 = 0.05$ which corresponds, from (5), to an increase in peak vorticity by a factor of four and a decrease in the diameter of the ring cross-section by a factor of about two. Although the ring is thinner initially, diffusion is correspondingly more rapid, and again only one clear passage of the rings is observed before they merge. However, as Maxworthy (1979) has observed it is the variation in behaviour of the vortex rings with Reynolds number that is important. And only as the Reynolds number increases substantially can we expect to observe the leapfrogging behaviour that may be realised with thin rings in an inviscid fluid.

To verify the above we have obtained numerical solutions which correspond to Reynolds numbers 500, 1000, 2000, 3000 and 4000 with $r_0 = 0.34$, $z_1 = 0.1$, $z_2 = 0.25$ and $t_0 = 0.05$ in all cases. The mesh sizes remained unchanged, as did the computational domain, except for the highest Reynolds number for which $0 \leqslant z \leqslant 3$ with δr, δz reduced by 2/3. The initial spacing of
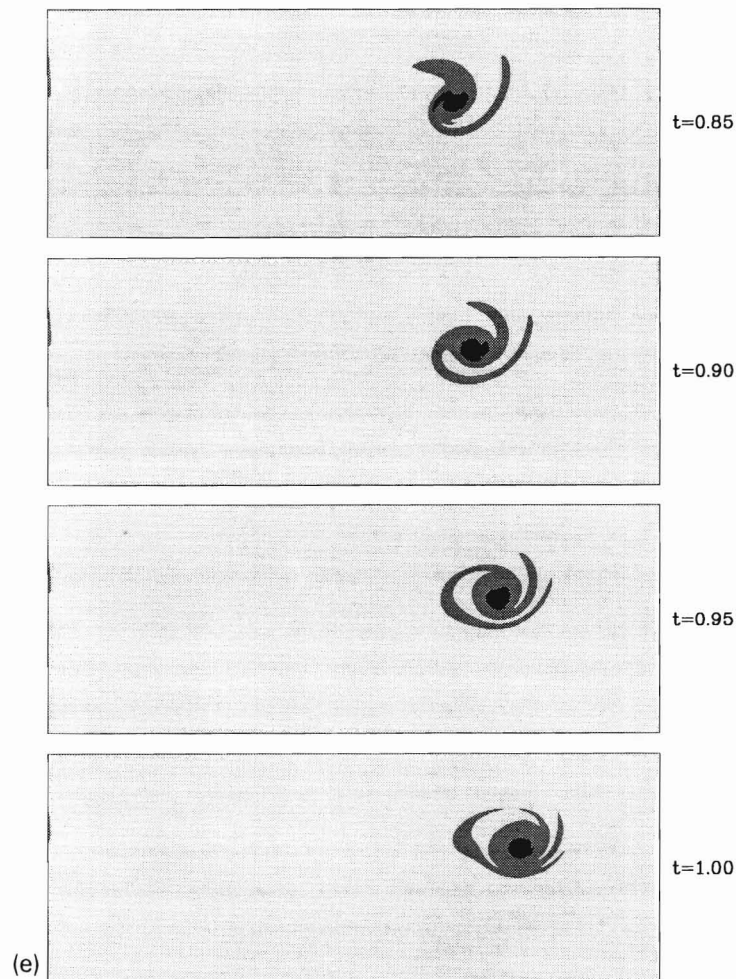
t=0.85

t=0.90

t=0.95

t=1.00

(e)

Fig. 1 (continued).

the rings is now less than for the results obtained at Re = 1000, described above, in order that the phenomena we describe be accommodated within our computational domain. And we note that as the initial separation increases, so the number of clear passes will decrease, since diffusion of vorticity will take place over a longer period of time before the first pass is made. With the aid of computer animation we have estimated the number of clear passes achieved by the rings, and the time taken for them to merge. The results are shown in Table 1. It is clearly demonstrated, that as the Reynolds number increases, so the behaviour of the vortex rings approaches that predicted by thin-ring inviscid-flow theory.

Consider now in detail the results of our investigation for Re = 2000 which are shown in fig. 1. Each frame of fig. 1 shows a lower cross-section of the two vortex rings. The upper boundary of each frame is the axis of symmetry, and the lower boundary corresponds to $r = 0.75$, whilst the two vertical boundaries represent the end walls $z = 0, 2.0$ of the computational domain. We see that at time $t = 0.2$ the rearward ring has passed successfully through the forward ring, and that by $t = 0.3$ the two rings have, essentially, changed places. During this time there has been some
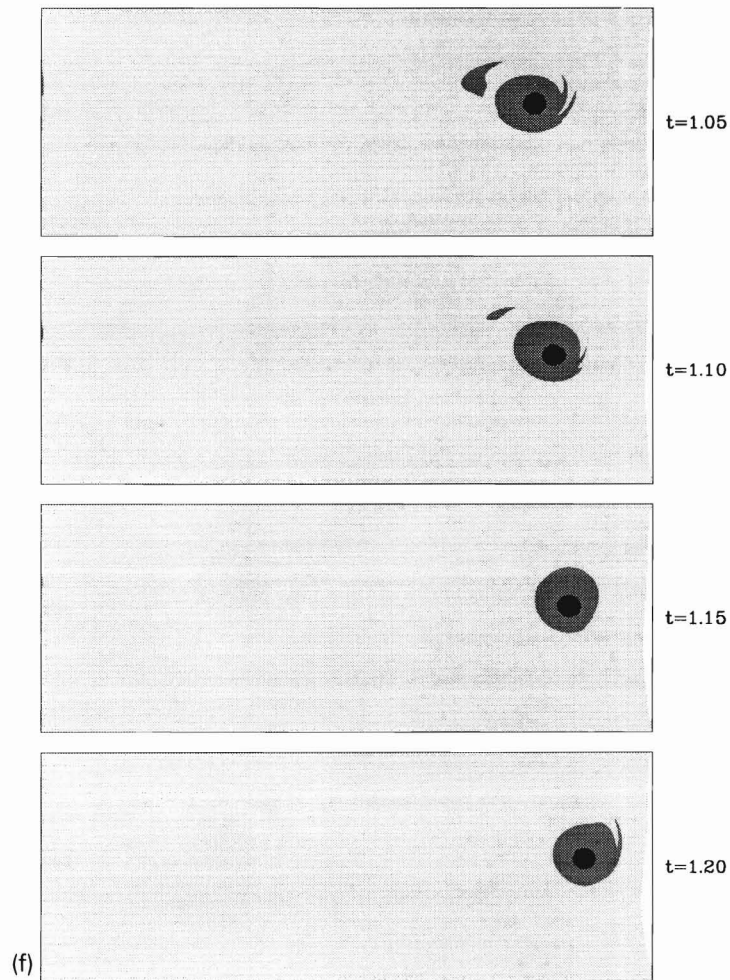
t=1.05

t=1.10

t=1.15

t=1.20

(f)

Fig. 1 (continued).

diffusion of vorticity but virtually no distortion of the relatively thin vortex cores. The process repeats itself with a second successful passage completed at $t = 0.45$, and the rings in their original relative position by $t = 0.55$. However, during the later stages of this second passage we see that the effects of diffusion are such that the two rings are already beginning to merge. And although a third passage appears to have been completed by $t = 0.7$, it is difficult to see how this could have been visualized in an experiment of the kind carried out by Yamada and Matsui. Beyond $t = 0.7$ the merging process is rapidly completed such that for times in excess of about 1.0 there is a single, rather diffuse, vortex ring. We have continued the calculation to show the effect of the interaction of the vortex ring with the end wall $z = 2.0$. Results by Orlandi, which are reproduced by Shariff and Leonard (1992), show similar features to those observed here. As the vortex ring approaches the end wall a thin boundary layer is created at it, within which the vorticity has opposite sign from that in the core of the ring. As the ring advances towards the end wall the effect of the boundary is to increase the diameter of the ring which itself sweeps up vorticity from the wall to create separation of the boundary layer, and indeed for $t = 1.35, 1.40$ there is evidence of reversed
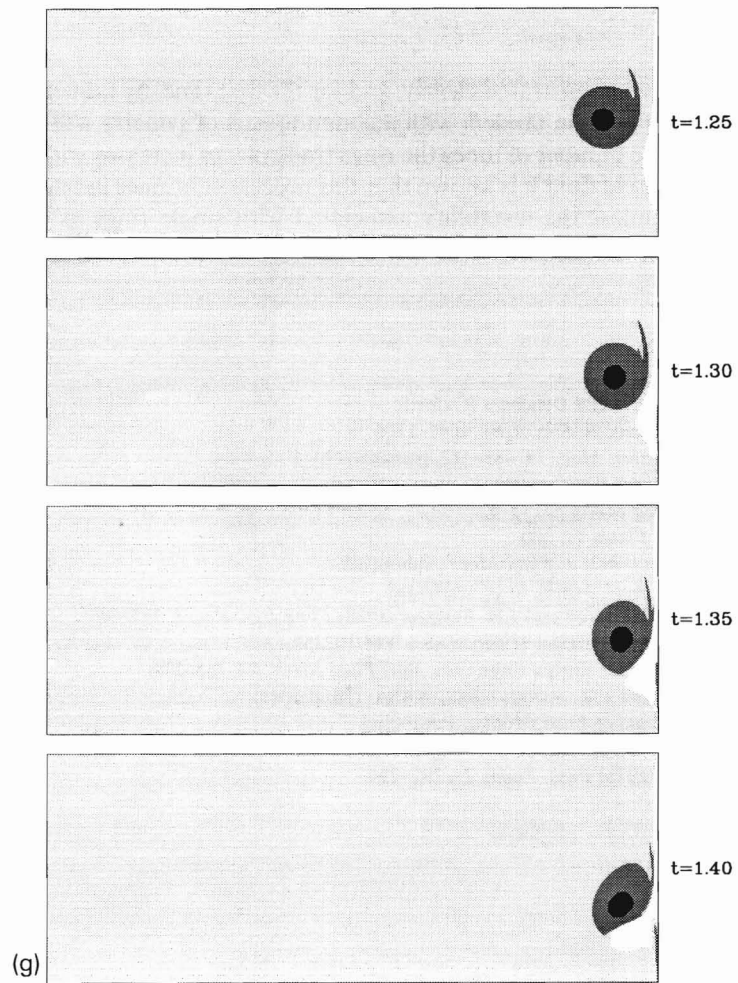
t=1.25

t=1.30

t=1.35

t=1.40

(g)

Fig. 1 (continued).

Table 1

Results from computer animation

Re	No. of passes	Time to merger
500	1	0.41
1000	2	0.58
2000	3	0.88
3000	4	1.16
4000	5	1.45

flow in the separation bubble. We do not present results for $t > 1.4$. But we can report that for this particular example the wall vorticity wraps itself around the core of the vortex ring and there is a gradual decay due to diffusion, with little further expansion of the ring.

3. Conclusions

In this paper we have demonstrated clearly, and for the first time, by numerical simulation that identical viscous vortex rings in tandem with a common axis of symetry will perform a leapfrogging motion in which the number of times the rings trade places increases with Reynolds number. For thin rings in an inviscid fluid it is known that this process continues indefinitely. Furthermore we have shown that, unless the instability associated with single rings at very high Reynolds numbers destroys them, the two rings will eventually merge into, and propagate as, a single ring.

References

Acheson, D. (1990) *Elementary Fluid Dynamics* (Oxford).
Dyson, F.W. (1893) *Philos Trans. R. Soc. A184*, 1041–1106.
Helmholtz, H. von (1858) *Philos. Mag. 33*, 485–512, translated by P.G. Tait.
Lamb, H. (1932) *Hydrodynamics* (Cambridge).
Maxworthy, T. (1972) *J. Fluid Mech. 51*, 15–32.
Maxworthy, T. (1979) *Phys. Fluids 22*, 200.
Riley, N. and P.D. Weidman (1993) *J. Fluid Mech.*, submitted.
Riley, N. (1993) *Q. J. Mech. Appl. Math.*, to be published.
Shariff, K., A. Leonard, N.J. Zabusky and J.H. Ferziger (1988) *Fluid Dyn. Res. 3*, 337–343.
Shariff, K., A. Leonard and J.H. Ferziger (1989) NASA TM-102257.
Shariff, K. and A. Leonard (1992) Vortex rings, *Ann. Rev. Fluid Mech. 24*, 235–279.
Stanaway, S., B.J. Cantwell and P.R. Spalart (1988) NASA TM-101041.
Van Dyke, M. (1982) *An Album of Fluid Motion* (Parabolic).
Woods, L.C. (1954) *Aero. Q. 5*, 176–182.
Yamada, H. and T. Matsui (1978) *Phys. Fluids 21*, 292–294.